Correction d'examen "Fonction à variable complexe"

(08 points).

Exercice n°1

$$* \quad (1+i)^i = e^{i\ln(1+i)} = e^{i[\ln|1+i| + i\arg(1+i)]}$$

$$= e^{i(\ln\sqrt{2} + i(\frac{\pi}{4}+2k\pi))}$$

$$= e^{i\ln\sqrt{2} - (\frac{\pi}{4}+2k\pi)}.$$

$$= e^{-(\frac{\pi}{4}+2k\pi)}\left(\cos(\ln\sqrt{2}) + i\sin(\ln\sqrt{2})\right).$$

1,5

$$* \quad ch(1+\tfrac{\pi}{2}i) = ch(1)\,ch(\tfrac{\pi}{2}i) + sh(1)\,sh(\tfrac{\pi}{2}i)$$

$$= ch1\cos\tfrac{\pi}{2} + i\,sh(1)\sin(\tfrac{\pi}{2})$$

$$= i\,sh(1) = i\left(\frac{e-e^{-1}}{2}\right).$$

1,5

$(E_1) : z^2 - 4z + 5 = 0$; $\Delta = (-4)^2 - 4(1)(5) = -4 = (2i)^2$

$z' = \dfrac{4-2i}{2} = 2-i$; $z'' = \dfrac{4+2i}{2} = 2+i$.

1,5

$(E_2) : -3z + i\bar{z} = -19 + 17i$; on pose $z = x+iy$.

2

$(E_2) \Leftrightarrow -3x - 3iy + ix + y = -19 + 17i$

$\Leftrightarrow \begin{cases} y - 3x = -19 \\ x - 3y = 17 \end{cases} \Rightarrow \begin{cases} x = 5 \\ y = -4 \end{cases} \Rightarrow z = 5 - 4i.$

$(E_3) : e^{-z+2} = 1-i$

1,5

$(E_3) \Leftrightarrow -z+2 = \ln(1-i)$

$\Rightarrow -z+2 = \ln(\sqrt{2}) + i(-\frac{\pi}{4} + 2k\pi)$

$\Rightarrow z = 2 - \ln(\sqrt{2}) - i(-\frac{\pi}{4} + 2k\pi).$

$\Rightarrow z = 2 - \ln(\sqrt{2}) + \frac{\pi}{4}i - 2k\pi i$; $k \in \mathbb{Z}$.

Exercice n°2 : Soient $z=x+iy$ et $u(x,y)=\frac{\sin x}{e^{y}}+x^2-y^2$. (6 points)

1) u est harmonique $\Leftrightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

$u(x,y)=\sin x\cdot e^{-y}+x^2-y^2$

$\frac{\partial u}{\partial x}(x,y)=\cos x\, e^{-y}+2x \Rightarrow \frac{\partial^2 u}{\partial x^2}(x,y)=-\sin x\cdot e^{-y}+2$. (2)

$\frac{\partial u}{\partial y}(x,y)=-\sin x\, e^{-y}-2y \Rightarrow \frac{\partial^2 u}{\partial y^2}(x,y)=\sin x\, e^{-y}-2$.

on a : $\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}=0$ alors u est harmonique.

2) f est holomorphe $\Leftrightarrow \begin{cases} \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \end{cases}$ (conditions de Cauchy-Riem)

$\frac{\partial v}{\partial y}(x,y)=\frac{\partial u}{\partial x}(x,y)=\cos x\cdot e^{-y}+2x$.

alors $v(x,y)=\int(\cos x\, e^{-y}+2x)\,dy=-\cos x\, e^{-y}+2xy+c(x)$ ...... (3)

donc $\frac{\partial v}{\partial x}(x,y)=\sin x\, e^{-y}+2y+c'(x)$. ...... (1) (2,5)

d'autre part on $\frac{\partial v}{\partial x}(x,y)=-\frac{\partial u}{\partial y}=\sin x\, e^{-y}+2y$ ...... (2)

de (1) et (2), on deduit que $c'(x)=0 \Rightarrow c(x)=c$. $(c\in\mathbb{R})$

d'après (3), on conclut que $v(x,y)=-\cos x\, e^{-y}+2xy+c$ $(c\in\mathbb{R})$.

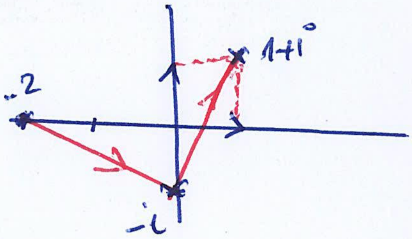
3) On a : $f'(z)=\frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y)$.

$=\cos x\, e^{-y}+2x+i(\sin x\, e^{-y}+2y)$. (1,5)

$=e^{-y}(\cos x+i\sin x)+2(x+iy)$.

$=e^{-y+ix}+2(x+iy)=e^{i(x+iy)}+2z=e^{iz}+2z$.

Exercice 3. (6 points) ; $z_A=-2$; $z_B=-i$; $z_C=1+i$



1) ($\mathcal{C}_1$) est le segment [AB].

$\gamma_1(t)=(1-t)z_A+tz_B$; $t\in[0,1]$

$=(1-t)(-2)+(-i)t$

$\gamma_1(t)=(2-i)t-2 \Rightarrow \gamma'(t)=(2-i)$.

on pose $z=\gamma(t)\Rightarrow dz=\gamma'(t)\,dt$; alors

$$\int_{\mathcal{C}_1}(z-1)\,dz=\int_0^1\big((2-i)t-3\big)(2-i)\,dt$$

$$=(2-i)\left[\frac{2-i}{2}t^2-3t\right]_0^1=(2-i)\left(\frac{2-i}{2}-3\right)=\frac{(2-i)(-4-i)}{2}=\frac{-9}{2}+i$$

0,75

2

($\mathcal{C}_2$) est le segment [BC].

$\gamma_2(t)=(1-t)z_B+tz_C$, $t\in[0,1]$.

$=(1-t)(-i)+t(1+i)$.

$\gamma_2(t)=(1+2i)t-i\Rightarrow\gamma_2'(t)=(1+2i)$.

$$\int_{\mathcal{C}_2}(z-1)\,dz=\int_0^1\big[(1+2i)t-(1+i)\big](1+2i)\,dt$$

$$=(1+2i)\left[\frac{1+2i}{2}t^2-(1+i)t\right]_0^1=(1+2i)\left(\frac{1+2i}{2}-1-i\right)=(1+2i)\left(\frac{-1}{2}\right)$$

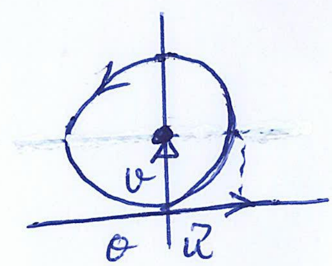
0,75

$$=-\frac{1}{2}-i.$$

alors $\int_{\mathcal{C}}(z-1)\,dz=\int_{\mathcal{C}_1}(z-1)\,dz+\int_{\mathcal{C}_2}(z-1)\,dz=-5$

0,5

2) ($\mathcal{C}$) est le cercle de centre $\Omega(0,1)$ ~~et de~~ rayon $R=1$



$\gamma(t)=z_\Omega+re^{it}$; $t\in[0,2\pi]$

1,5

$\gamma(t)=i+e^{it}\Rightarrow\gamma'(t)=ie^{it}$.

$$\int_{\mathcal{C}}(z-i)^2\,dz=\int_0^{2\pi}(\cancel{i}+e^{it}-i)^2\,ie^{it}\,dt=i\int_0^{2\pi}e^{i3t}\,dt=i\left[\frac{1}{3i}e^{i3t}\right]_0^{2\pi}$$

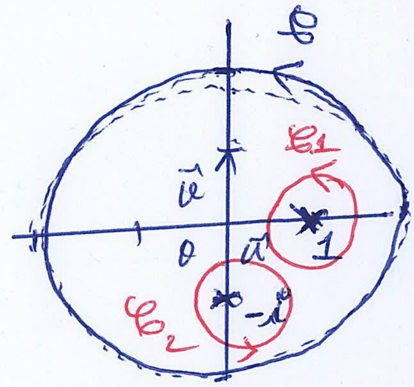
$$=\frac{1}{3}\left[e^{i6\pi}-1\right]=0$$

*

* methode 2. $\int_{\mathcal{C}} (z-i)^2 dz$, $f(z)=(z-i)^2$ est holomorphe sur $\mathbb{C}$

alors f est holomorphe sur $(\mathcal{C})$, donc d'après

Théorème de Cauchy: $\int_{\mathcal{C}} f(z)dz = 0$.



3) On utilise les formules de Cauchy:

d'après le Théorème de Cauchy, on a.

$$\oint_{\mathcal{C}} f(z)dz = \oint_{\mathcal{C}_1} f(z)dz + \oint_{\mathcal{C}_2} f(z)dz.$$

$$\oint_{\mathcal{C}_1} f(z)dz = \oint_{\mathcal{C}_1} \frac{\frac{e^z}{(z+i)^2}}{z-1}dz = \oint_{\mathcal{C}} \frac{G(z)}{z-1}dz = 2\pi i\, G(1) = \frac{e}{(1+i)^2} 2\pi i$$

1

$$= 2\pi i \frac{e}{2i} = e\pi.$$

$$\oint_{\mathcal{C}_2} f(z)dz = \oint_{\mathcal{C}_2} \frac{\frac{e^z}{z-1}}{(z+i)^2}dz = \oint_{\mathcal{C}_2} \frac{H(z)}{(z+i)^2}dz = 2\pi i\, H'(-i).$$

$$H(z) = \frac{e^z}{z-1} \Rightarrow H'(z) = \frac{e^z(z-1)-e^z}{(z-1)^2} = \frac{(z-2)e^z}{(z-1)^2}$$

$$H'(-i) = \frac{(-i-2)e^{-i}}{(-1-i)^2} = \frac{-(2+i)e^{-i}}{2i}.$$

1

donc $\oint_{\mathcal{C}_2} f(z)dz = -\pi(2+i)e^{-i}$.

0,5

On resulte que $\oint_{\mathcal{C}} f(z)dz = e\pi - \pi(2+i)e^{-i}$.